A simplified analysis of spherical and cylindrical blast waves

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Investigations into the behaviour of the gas flow behind spherical or cylindrical blasts have shown that secondary shocks arise within the original detonation gases. The secondary shock, at first weak, is carried outward with the expanding gases. Subsequently it strengthens and bends back toward the origin, arriving there with high intensity.

By using some recently developed techniques in shock dynamics and extending them where necessary, a theory is developed by which the motion of the main shock wave, as well as the formation and subsequent motion of the secondary shock, are given by explicit formulae. In addition, a method for determining, also by explicit formulae, the location of the contact surface between the detonation gases and the outside atmosphere is given. The results of a specific problem, which has been solved by numerically integrating the total equations of motion, and has also been checked experimentally, are compared with the results of the present theory.

1. Introduction

Recent theoretical and experimental studies of spherical and cylindrical blast waves (Boyer 1960; Brode 1957, 1959; Shardin 1954; Wecken 1950) enable us to give the following qualitative description of these processes. Assume that at time $t = 0$ a gas sphere (or cylinder) of radius $x = x_0$ under high internal pressure, $p = p_4$, is in a still air atmosphere at pressure $p = p_0$ ($p_4 \gg p_0$). The names 'gas' and 'air', respectively, will henceforth be used to describe the fluids initially within and outside this sphere. For $t > 0$, an equalization (explosion) process takes place. At the initial point $t = 0$, $x = x_0$, the region $x \geqslant 0$, $t > 0$ can be separated into five domains (figure 1): (0) undisturbed air; (1) air which has been overtaken by the main blast wave; (2) nearly uniform region outside the main expansion; (3) gas in the main expansion region; (4) gas not yet disturbed by the expansion. The gas in region (2) and the air in region (1) are separated by a contact front. Also, a secondary shock develops and separates regions (3) and (2).

The interesting phenomenon of a secondary shock appears only in spherical (and cylindrical) flows and does not arise in one-dimensional shock-tube studies. For the latter the main shock and the expansion come into an instantaneous equilibrium, being separated by a region of uniform pressure and velocity. The physical reason for the secondary shock formation is that the high-pressure gas, upon passing through a spherical rarefaction wave, must expand to lower pressures than those reached through an equivalent one-dimensional expansion. This is due to the increase of volume. Because of this 'over-expansion', the pressures at the tail of the rarefaction wave are lower than the pressures transmitted back by the main shock, and a compression, or second shock, must be inserted to connect the two phases. By mathematical reasoning the situation is further clarified: the centred expansion is accomplished through the negative characteristics (of slope $dx/dt = u-a$, where u is particle velocity and a is sound speed). The characteristics at the head of the wave point in the decreasing x-direction (subsonic flow), and they then fan around to the increasing x-direction as the velocity increases. Negative characteristics are also reflected from the main shock, but as the shock expands it becomes weaker and these character-

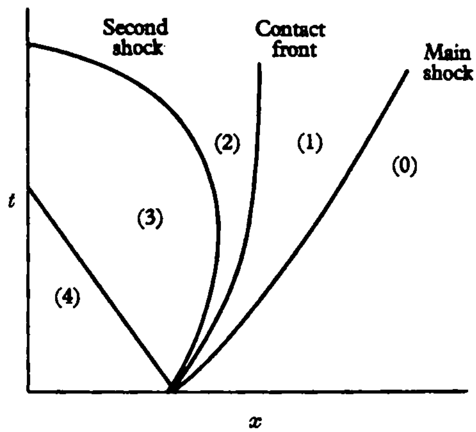

FIGURE 1. Explosion flow diagram. The indicated regions are: (0) undisturbed air; (1) air overtaken by main blast wave; (2) gas which has passed through secondary shock; (3) expanding gas; (4) undisturbed gas.

istics incline more and more toward the decreasing x-direction. Hence, characteristics of the same family, but arising from two different sources, will tend to intersect. This is prevented by the secondary shock. A similar situation arises in one-dimensional flows when a shock moving down a uniform tube meets an area change (Friedman 1960). The shock strength is attenuated, and characteristics reflected from the shock will be bent. They will meet the characteristics of the same family which originated in the uniform section of the tube unless a secondary shock is inserted.

This reasoning in terms of characteristics leads directly to a method for determining explicitly the path of the secondary shock. It is easily shown (Courant & Friedrichs 1948, p. 159) that the slope of a weak shock, at each point, is nearly equal to the average of the slopes of the incoming characteristics at that point. Using this fact, Whitham (1952) developed a technique for obtaining a differential equation for the shock path. This technique is used with the present problem to get the initial motion of the secondary shock.

As the secondary shock moves outward it strengthens, and the above 'weak shock' theory is no longer applicable. In order to follow further shock development, a relation obtained by Chisnell (1957) and amplified by Whitham (1958) is introduced. Chisnell considered a shock wave moving in a channel with varying cross-sectional area and derived an analytic expression which gives the relation between the channel area and shock strength. For the present analysis the channel area is identified with the shock surface area, and hence is related to the shock radius. Whitham simplified and extended Chisnell's analysis by showing that the 'shock-area' relation can be obtained by the following method: the differential relation which is to be satisfied by the flow quantities along a characteristic coming into a shock is applied to the flow quantities just behind the shock; then, if they are expressed in terms of the shock strength, a differential equation is obtained relating changes in shock strength and radius. This differential equation can be integrated, if the flow ahead of the shock is uniform, to give Chisnell's result. The inward-facing secondary shock of the present problem, however, has ahead of it the non-uniform expanding gas. Here the shock motion depends not only on its radius but also on the flow conditions immediately ahead of it. Whitham's derivation of the 'shock-area' relation is extended to include this situation. The secondary shock is therefore obtained in two segments which take into account its changing nature from very weak, initially, to quite strong when it finally implodes at the origin. When it is weak, the average slope technique will be used to describe its motion until the shock strength becomes too large. From this point on, the extended shock-area rule is used.

The determination of the main shock wave separating regions (0) and (1) is much simpler than that of the secondary shock. This is because the flow ahead of the main shock in region (0) is uniform, and direct application of the integrated form of the shock-area relation suffices to give the shock strength and location. It should be emphasized that the shock-area relation is expected to be a good approximation in this problem because the shock weakens primarily due to its increased surface area; in the one-dimensional case its strength would remain constant. This is in contrast to the Taylor blast-wave problem, for example, where the weakening effect of disturbances from behind is equally important.

Flow properties arising at the main shock are propagated back into the interior along the negative characteristics. These cross the contact front and are altered upon entering the gas. In order to determine the flow changes at the contact front it is necessary to determine the path of the front. Toward this end a differential equation, obtained in a similar manner to the one describing the weak shock path, has been derived, i.e., the motion of the contact front is prescribed by known flow properties on characteristics meeting it.

At some places in the present analysis the additional entropy changes above those in the one-dimensional problem will be neglected. This will not affect the calculations in the expansion region (3) since the expansion is isentropic, nor in the initial portions of region (2) since the flow here has passed through the weak section of the secondary shock. For determination of the shocks by application of the shock-area rule, entropy effects are correctly included in the shock conditions. It is seen then that the only place where neglect of entropy changes could seriously affect the calculations is in following the flow properties from the main shock through region (1) to the contact front.

The slope of the secondary shock, when it initially forms and moves outward, is determined by the slopes of the negative characteristics of regions (3) and (2) which meet it. In § 2 an approximate solution for the flow in the expansion region (3) is determined. Since the flow near the initial point can be accurately described by one-dimensional shock-tube theory, a solution in region (3) which includes three-dimensional effects is obtained by perturbing about the known one-dimensional solution. This improved solution is used to give negative characteristics in region (3), and these are in turn used for the determination of the initial motion of the secondary shock. In addition, this solution gives information about the incoming flow which is required for use with the shock-area rule when the secondary shock strengthens. The negative characteristics for region (2) originate at the main shock, move through region (1) and have a discontinuity in slope upon crossing the contact front. For determination of these characteristics, the main shock and negative characteristics reflected from it must first be found; this is accomplished in § 3. The contact front and its affect on characteristics crossing it is discussed in § 4. Here a differential equation is determined by which the path of the contact front is given in terms of known flow quantities. In § 5 the results of the previous sections are utilized to give the motion of the secondary shock. Here both the weak-shock theory and the extended shock-area rule are presented. A specific problem involving a spherical blast is treated in § 6, and the results are compared with other solutions to this problem.

As a consequence of the form of the shock-area relation, the x and t co-ordinates of the main shock are determined parametrically as functions of shock Mach number. Similarly, since the contact front and secondary shock are determined by extrapolation of the known flow conditions at the main shock, their co-ordinates are also determined as parametric functions of the main shock Mach number. Therefore, starting at the initial point with the known main shock Mach number, small incremental steps in this Mach number are made. Corresponding to each increment, a point on the main shock, contact front and secondary shock are determined. The present analysis will describe the motion of the secondary shock as it initially moves outward and until it finally implodes at the origin. Beyond that time, its motion cannot be given by this theory. Similarly, this time will roughly delineate the region of validity for the main-shock and contact-front approximations.

2. Expansion region

Immediately after detonation, the high-pressure gas in region (4) of figure 1 is penetrated by a centred expansion fan and moves rapidly outward. This expansion, region (3), is isentropic and satisfies the following flow equations written in characteristic form:

$$\left(\frac{1}{\gamma-1}a_t+\tfrac{1}{2}u_t\right)+(u+a)\left(\frac{1}{\gamma-1}a_x+\tfrac{1}{2}u_x\right)+\frac{nua}{2x}=0, \tag{1}$$

$$\left(\frac{1}{\gamma-1}a_t-\tfrac{1}{2}u_t\right)+(u-a)\left(\frac{1}{\gamma-1}a_x-\tfrac{1}{2}u_x\right)+\frac{nua}{2x}=0, \tag{2}$$

where $n = 1, 2$ for cylindrically symmetric flows and spherically symetricm flows, respectively, and where γ is the ratio of specific heats. The flow variables a and u, the sound speed and particle velocity, and also the independent variables x and t, the radial distance and time, are all assumed to have been made dimensionless as follows:

$$a = \bar{a}/a_0, \quad u = \bar{u}/a_0, \quad x = \bar{x}/x_0, \quad t = \bar{t}a_0/x_0.$$

Barred quantities are true physical variables, a_0 is the sound speed outside the blast wave in region (0) of figure 1, and x_0 is the initial blast radius.

An approximate solution to equations (1) and (2) is obtained by perturbing about a one-dimensional centred expansion wave, whose solution can be given explicitly (Courant & Friedrichs 1948, p. 104). This approximation holds exactly as the initial point $x = 1$, $t = 0$ is approached. We first define

$$\left.\begin{aligned} &u = u_1+u_2, \quad a = a_1+a_2, \\ &r = \frac{1}{\gamma-1}a_1+\tfrac{1}{2}u_1, \quad s = \frac{1}{\gamma-1}a_1-\tfrac{1}{2}u_1, \\ &R = \frac{1}{\gamma-1}a_2+\tfrac{1}{2}u_2, \quad S = \frac{1}{\gamma-1}a_2-\tfrac{1}{2}u_2, \end{aligned}\right\} \tag{3}$$

where the variables with subscript 1 are solutions to the one-dimensional centred expansion problem (equations (1) and (2) with $n = 0$); that is,

$$\left.\begin{aligned} &u_1 = 2\mu r+(1-\mu)\frac{x-1}{t}, \\ &a_1 = \mu\left(2r-\frac{x-1}{t}\right), \quad \mu = \frac{\gamma-1}{\gamma+1}, \quad r = \text{const.}, \\ &s = (1-2\mu)r-(1-\mu)\frac{x-1}{t}. \end{aligned}\right\} \tag{4}$$

Substituting (3) and (4) into (1) and retaining first-order terms, we obtain, as an equation for R,

$$R_t+(u_1+a_1)R_x+\frac{nu_1a_1}{2x} = 0,$$

or

$$\frac{dt}{t} = \frac{dx}{4\mu rt+(1-2\mu)(x-1)} = -\frac{2x\,dR}{ntu_1a_1}. \tag{5}$$

The first relation in (5) gives, for the curvilinear positive characteristics,

$$\frac{x-1}{t}-2r = -Kt^{-2\mu}, \quad \text{or} \quad a_1t^{2\mu} = K\mu, \tag{6}$$

where K is a constant for each characteristic.

Using (6) to eliminate x, we obtain the differential equation for R along the positive characteristics

$$\frac{dR}{dt} = -\frac{n\mu K[2r-(1-\mu)Kt^{-2\mu}]t^{-2\mu}}{2[1+t(2r-Kt^{-2\mu})]}. \tag{7}$$

It does not seem possible to integrate this equation for arbitrary μ. However, for certain specific values an integral in closed form is possible. The integration for $\mu = \frac{1}{6}$ and $\frac{1}{4}$, corresponding to $\gamma = \frac{7}{5}$ and $\frac{5}{3}$, can be carried out, but with the present approximate theory the added complexity of the exact integral does not seem justified. The expression on the right-hand side of (7) is therefore simplified by assuming the bracketed term in the denominator equal to 1. This implies that the characteristic remains close to the line $x = 1$, which is roughly correct except for very high intensity blast waves. On this assumption we have

$$\frac{dR}{dt} = -\frac{n\mu K}{2}[2r-(1-\mu)Kt^{-2\mu}]t^{-2\mu},$$

which can be integrated to give

$$R = -\frac{nK\mu t}{2}\left[\frac{2r}{1-2\mu}t^{-2\mu} - \frac{(1-\mu)K}{1-4\mu}t^{-4\mu}\right] + f(a_1 t^{2\mu}).$$

Eliminating K by means of (6), we get

$$R = \frac{nt}{2}\left[\frac{(1-\mu)}{\mu(1-4\mu)}a_1^2 - \frac{2r}{1-2\mu}a_1\right] + f(a_1 t^{2\mu}). \tag{8}$$

The arbitrary function $f(a_1 t^{2\mu})$, constant along each positive characteristic, and the constant r are determined by continuously connecting $(\gamma-1)^{-1}a + \frac{1}{2}u$ across the boundary characteristic between regions (3) and (4). Since we shall assume (4) to be a region of uniform flow with $u_4 = 0$ and a_4 constant, the boundary characteristic is

$$x - 1 = -a_4 t. \tag{9}$$

For this case we would have

$$\left.\begin{aligned} r &= \frac{1}{\gamma-1}a_4, \\ f(a_1 t^{2\mu}) &= -\frac{tn}{2}\left(\frac{a_1}{a_4}\right)^{1/2\mu}\left[\frac{1-\mu}{\mu(1-4\mu)}a_4^2 - \frac{2r}{1-2\mu}a_4\right]. \end{aligned}\right\} \tag{10}$$

The second equation in (10) is obtained by requiring R to vanish at the boundary between regions (3) and (4). It is clear that by properly choosing the arbitrary characteristic function $f(a_1 t^{2\mu})$ appearing in equation (8), a more general boundary condition could be satisfied. Combining (8) and (10) we have, for region (3),

$$\frac{1}{\gamma-1}a_3 + \tfrac{1}{2}u_3 = \frac{1}{\gamma-1}a_4 + tH(a_1), \tag{11}$$

with $$H(a_1) = \frac{n}{2}\left\{\frac{1-\mu}{\mu(1-4\mu)}\left[a_1^2 - \left(\frac{a_1}{a_4}\right)^{1/2\mu}a_4^2\right] - \frac{2r}{1-2\mu}\left[a_1 - \left(\frac{a_1}{a_4}\right)^{1/2\mu}a_4\right]\right\},$$

$$a_1 = \mu\left(2r - \frac{x-1}{t}\right).$$

The second-order terms in the negative characteristic approximation are obtained by substituting the relations (3) and (4) into the negative characteristic equation (2), the equation obtained for S being

$$tS_t + (x-1)S_x + [(2\mu-1)R + S] + \frac{nu_1 a_1 t}{2x} = 0, \tag{12}$$

where $R = tH(a_1)$ and H is defined in (11). Writing (12) in characteristic form, we have

$$\frac{dt}{t} = \frac{dx}{x-1} = \frac{dS}{-[(2\mu-1)R+S] - \dfrac{nu_1 a_1 t}{2x}}. \tag{13}$$

The first of equations (13) gives, for the characteristics, the lines $x-1 = Lt$, with L constant along each characteristic. Along the characteristics, (13) also gives

$$\frac{d}{dt}(tS) = (1-2\mu)R - \frac{nu_1 a_1 t}{2(1+Lt)} = (1-2\mu)tH(a_1) - \frac{nu_1 a_1 t}{2(1+Lt)}. \tag{14}$$

Since u_1 and a_1 are constant along the characteristic lines, (14) can be integrated immediately to give

$$S = (1-2\mu)\frac{t}{2}H(a_1) - \frac{nu_1 a_1}{2L}\left[1 - \frac{\log(1+Lt)}{Lt}\right].$$

Letting $L = (x-1)/t$ and using equation (4), we have

$$\frac{1}{\gamma-1}a_3 - \tfrac{1}{2}u_3 = (1-2\mu)r - (1-\mu)\frac{x-1}{t} + \frac{(1-2\mu)}{2}tH(a_1) - \frac{nu_1 a_1 t}{2(x-1)^2}[x-1-\log x]. \tag{15}$$

The equation for the negative characteristics can now be determined. The lowest-order approximation for this is $x = 1+Lt$; however, by use of the above results, a more exact solution is obtained. Taking a proper linear combination of (11) and (15), we get, for the negative characteristic slope,

$$u_3 - a_3 = \frac{x-1}{t} + \frac{1-2\mu}{2-2\mu}tH(a_1) + \frac{nu_1 a_1 t}{2(1-\mu)(x-1)^2}[x-1-\log x]. \tag{16}$$

To facilitate integration, the right-hand side of (16) is simplified by assuming that $x = 1+Lt$ and that u_1, a_1 and L are constant along the negative characteristics. This gives

$$\frac{dx}{dt} = L + \frac{1-2\mu}{2-2\mu}tH(a_1) + \frac{nu_1 a_1 t}{2(1-\mu)L}\left[1 - \frac{\log(1+Lt)}{Lt}\right],$$

$$x \simeq 1 + Lt + \frac{(1-2\mu)t^2}{4(1-\mu)}H(a_1) + \frac{nu_1 a_1}{2(1-\mu)L}\left[1 - \frac{\log(1+Lt)}{Lt}\right]\frac{t}{2},$$

or, if we let $L = (x-1)/t$ in the last term,

$$x \simeq 1 + Lt + \frac{(1-2\mu)t^2}{4(1-\mu)}H(a_1) + \frac{nu_1 a_1 t^2}{4(1-\mu)(x-1)}\left[1 - \frac{\log x}{x-1}\right], \tag{17}$$

where $H(a_1)$ is given in equation (11). Since the logarithmic term in (16) does not vary much, an approximate integration was used to obtain (17).

Equations (11), (15), (16) and (17) will be used for describing the flow in region (3).

3. Main shock and region (1)

The characteristic equations (1) and (2), written with pressure and velocity as dependent variables, are

$$p_t + \rho a u_t + (u+a)(p_x + \rho a u_x) + \frac{\rho a^2 u n}{x} = 0, \tag{18}$$

$$p_t - \rho a u_t + (u-a)(p_x - \rho a u_x) + \frac{\rho a^2 u n}{x} = 0. \tag{19}$$

When written in this form, the equations are valid whether or not the flow is isentropic.

Since the positive characteristics are the ones coming into the main shock, equation (18) is used in an application of Whitham's rule (given in § 1) for the determination of the shock motion. We write (18) in the form

$$dp + \rho a\,du = -\frac{\rho a^2 u n}{u+a}\frac{dx}{x}. \tag{20}$$

At the shock the Hugoniot relations are

$$\left.\begin{aligned} u &= \frac{2a_0}{\gamma+1}(M - M^{-1}), \quad p = \frac{p_0}{\gamma+1}(2\gamma M^2 - \gamma + 1), \\ \rho &= \frac{\rho_0(\gamma+1)M^2}{(\gamma-1)M^2+2}, \quad a^2 = \frac{a_0^2(2\gamma M^2-\gamma+1)\{M^2(\gamma-1)+2\}}{(\gamma+1)^2 M^2}, \end{aligned}\right\} \tag{21}$$

where $M = U/a_0$, U is the shock velocity, and subscript 0 refers to values ahead of the shock (region (0) in figure 1). Substituting these relations into (20) we obtain a differential relation between the shock Mach number and its radius x_m:

$$\begin{aligned} -n\frac{dx_m}{x_m} = dM\bigg\{&\frac{4M}{2\gamma M^2-\gamma+1} + \frac{2(M^2+1)}{M\sqrt{\{[2\gamma M^2-\gamma+1][M^2(\gamma-1)+2]\}}} \\ &+ \frac{2M}{M^2-1}\sqrt{\left\{\frac{(\gamma-1)M^2+2}{2\gamma M^2-\gamma+1}\right\}} + \frac{M^2+1}{M(M^2-1)}\bigg\}. \end{aligned} \tag{22}$$

This can be integrated to give

$$\begin{aligned} (x_m)^n &\left[\frac{\{2\gamma M^2-\gamma+1\}^{\frac{1}{2}} - \{(\gamma-1)M^2+2\}^{\frac{1}{2}}}{M}\right]^2 [\{(\gamma-1)(2\gamma M^2-\gamma+1)\}^{\frac{1}{2}} \\ &- \{2\gamma[(\gamma-1)M^2+2]\}^{\frac{1}{2}}]^{\sqrt{\{2\gamma/(\gamma-1)\}}}(2\gamma M^2-\gamma+1)^{1/\gamma} \\ &\times \exp\left[\frac{1}{\{2(\gamma-1)\}^{\frac{1}{2}}}\sin^{-1}\left\{\frac{2(2\gamma M^2-\gamma+1)-(\gamma-1)[(\gamma-1)M^2+2]}{M^2(\gamma+1)^2}\right\}\right] = \text{const.} \end{aligned} \tag{23}$$

Using the dimensionless co-ordinates introduced in (3), we relate t_m and M, along the shock, by

$$M\,dt_m = dx_m, \tag{24}$$

with dx_m defined in equation (22).

The motion of the main shock as a function of its Mach number is given by equations (23) and (24) in conjunction with an initial condition specifying its position and Mach number at a certain time, ordinarily the instant of detonation. For both spherical and cylindrical blast waves, the initial motion is insensitive to the geometrical situation, and the initial shock Mach number can be determined by use of one-dimensional shock-tube theory. Specifically, if the gas inside the sphere (or cylinder) were steady with pressure p_4 and sound speed a_4, and the outside atmosphere were steady at p_0 and a_0, the one-dimensional theory would give the following relation, which the initial shock Mach number must satisfy:

$$\frac{p_4}{p_0}\left\{1-\frac{\mu a_0}{a_4}(M-M^{-1})\right\}^{(\mu+1)/\mu} = (1+\mu)M^2-\mu. \tag{25}$$

It will be seen that the only flow property required in region (1) is the position of the negative characteristics, reflected from the main shock. Since the contact front follows closely behind the main shock, region (1) is small, and it suffices to approximate the negative characteristics there by straight lines. Essentially, this approximation neglects entropy and three-dimensionality. Effects of the changing shock strength, due to expansion, are propagated into region (1) by adjustments of the slope of each characteristic to be that value of $u-a$ where the characteristic meets the shock. The characteristics are therefore

$$x = x_m + w_1(M)(t-t_m), \tag{26}$$

with $w_1(M) = u-a$ evaluated at the point (x_m, t_m) on the main shock.

Equations (23), (24) and (26) define the main shock and negative characteristics behind the shock as functions of the shock Mach number M.

4. Contact front

The negative characteristics arising at the main shock propagate through region (1) and meet the contact front separating the gas in region (2) from the air in region (1). At this front the characteristic suffers an abrupt change in slope and continues into region (2). In order to determine this slope discontinuity, the location of the contact front and its point of intersection with the characteristic must be known. A technique for obtaining these is now developed.

Since the contact front moves with the local particle velocity u, it will be met by positive characteristics, with slope $u+a$, from region (2) and negative characteristics, with slope $u-a$, from region (1). The positive characteristic values are obtained from equation (11). Although this equation was derived to give $(\gamma-1)^{-1}a+\frac{1}{2}u$ in region (3), we shall assume it to be applicable in region (2), since the secondary shock which separates regions (2) and (3) is weak where this approximation is made. Let Q_2, Q_1 and w_2, w_1 denote $(\gamma-1)^{-1}a+\frac{1}{2}u$ and $u-a$ in regions (2) and (1) respectively. We have, due to continuity of velocity across the contact front,

$$\left.\begin{aligned} Q_2+\frac{1}{\gamma-1}w_2 &= Q_1+\frac{1}{\gamma-1}w_1, \\ \text{or}\qquad w_2-w_1 &= -(\gamma-1)(Q_2-Q_1). \end{aligned}\right\} \tag{27}$$

On a particle path, we have the relation between pressure and sound speed

$$\frac{p}{p_i} = \left(\frac{a}{a_i}\right)^{2\gamma/(\gamma-1)},$$

where subscript i stands for the initial state. Consequently, due to continuity of pressure across the contact front,

$$\left.\begin{aligned} \left(\frac{a}{a_i}\right)_2 &= \left(\frac{a}{a_i}\right)_1 \\ \text{or}\qquad 2Q_2-w_2 &= \frac{a_{i2}}{a_{i1}}(2Q_1-w_1). \end{aligned}\right\} \tag{28}$$

(It should be noted that in the derivation of equations (27) and (28) the assumption has been made that γ, the ratio of specific heats, has the same magnitude in both regions (1) and (2). The extension to the case of different γ is direct but not very simple.) We obtain the negative characteristic slopes for region (2) in terms of known quantities by eliminating Q_1 from equations (27) and (28): thus,

$$w_2 = \frac{\left(\frac{\gamma+1}{\gamma-1}\right)\frac{a_{i2}}{a_{i1}}w_1 + 2\left(1-\frac{a_{i2}}{a_{i1}}\right)Q_2}{1+\left(\frac{2}{\gamma-1}\right)\frac{a_{i2}}{a_{i1}}}. \tag{29}$$

Evaluation of (29) is not direct, since w_1 is known only as a function of main shock Mach number (equation (26)) and Q_2 is known only as a point function of x and t (equation (11)). We must determine, therefore, the (x,t)-co-ordinates of the contact front in terms of the parameter M.

First, a is expressed in terms of w_1 and Q_2. Since velocity is continuous across the contact front, we can write

$$2Q_2 - w_1 = \frac{2}{\gamma-1}a_2 + a_1,$$

and, combining this with the first of equations (28),

$$a_1 = \frac{2Q_2 - w_1}{1+\left(\frac{2}{\gamma-1}\right)\frac{a_{i2}}{a_{i1}}}. \tag{30}$$

Next, let $x_c = C(t_c)$ be the equation of the contact front. The negative characteristic given in equation (26) meets the contact front at the point (x_c, t_c). Hence, at this point,

$$C(t_c) = x_m + w_1(M)(t_c - t_m).$$

Differentiating with respect to t_c, we get

$$\frac{dC}{dt_c} = w_1 + \frac{dM}{dt_c}\{x'_m + w'_1(t_c - t_m) - w_1 t'_m\}.$$

Here primes indicate differentiation with respect to M. As the contact front moves with local particle velocity, it follows that

$$\frac{dC}{dt_c} = \frac{dx_c}{dt_c} = u.$$

Combining the two above equations, using $w = u - a$, we have

$$a_1 = \frac{dM}{dt_c}\{x'_m + w'_1(t_c - t_m) - w_1 t'_m\}. \tag{31}$$

Eliminating a_1 from (30) and (31), we obtain a differential equation relating t_c and M:

$$\frac{dt_c}{dM} = \frac{\left\{1+\left(\frac{2}{\gamma-1}\right)\frac{a_{i2}}{a_{i1}}\right\}\{x'_m + w'_1(t_c - t_m) - w_1 t'_m\}}{2Q_2 - w_1}. \tag{32}$$

Equation (32), together with

$$x_c = x_m + w_1(M)(t_c - t_m), \tag{33}$$

define the co-ordinates of the contact front (x_c, t_c) as functions of the parameter M. With this result we have directly, for the negative characteristics in region (2),

$$x = x_c + (t - t_c)\, w_2, \tag{34}$$

where t_c and x_c are given by (32) and (33), and w_2 is given by (29).

5. Secondary shock

We now have enough information about the flow to fit the secondary shock between regions (3) and (2). The negative characteristics arising in the expansion region (3) fan into the increasing x-direction, while those arising at the main shock tend to point more and more toward the decreasing x-direction. The latter is due to the fact that the main shock weakens as it expands, causing the characteristic slope $u_1 - a_1$ to decrease.

The secondary shock must be inserted so that the shock relations are satisfied; these are approximated, for a weak shock, by requiring its slope at each point to equal the average of the slopes of the characteristics meeting it. Since we have the equations, (17) and (34), describing the characteristics which meet the shock, we can, using a technique developed by Whitham (1952), obtain the shock motion.

The characteristics which arise in the expansion fan and at the main shock are respectively

$$\left.\begin{aligned} x &= 1 + Lt + f(x, t), \\ x &= x_c + w_2(t - t_c), \end{aligned}\right\} \tag{35}$$

where $f(x, t)$ represents terms appearing in equation (17), and x_c, t_c, and w_2 are functions of main shock Mach number M as given in equations (33), (32) and (29). Assuming the secondary shock path to be given by

$$x_s = 1 + S(t_s),$$

we have along the shock path

$$\left.\begin{aligned} S(t_s) &= w_2(t_s - t_c) + x_c - 1 \\ &= Lt_s + f\{x_c + (t_s - t_c)\, w_2, t_s\}. \end{aligned}\right\} \tag{36}$$

Expressing the shock slope as the average of the characteristic slopes as obtained from (35), we obtain

$$\frac{dS}{dt_s} = \frac{1}{2}\left(w_2 + \frac{L + f_t}{1 - f_x}\right). \tag{37}$$

Letting the parameters M and L and the variable x_s all be dependent on t_s, we can obtain another representation for dS/dt_s, using (35) and (36); with primes representing differentiation with respect to M, this is

$$\frac{dS}{dt_s} = \frac{1}{2}\left(w_2 + \frac{dM}{dt_s}\,[x_c' + w_2'(t_s - t_c) - w_2 t_c'] + \frac{L + f_t}{1 - f_x} + \frac{t(dL/dt)}{1 - f_x}\right).$$

When combined with (37), this equation yields

$$t\,\frac{dL}{dM} = -(1 - f_x)\,\{x_c' + w_2'(t_s - t_c) - w_2 t_c'\}. \tag{38}$$

Eliminating L from (38) by use of (36), we obtain, after working through some algebra and integrating with respect to M, a relation between t_s and M:

$$[x_c - 1 - t_c w_2]^2 = t_s \int_{M_i}^{M} [x_c - 1 - t_c w_2] \left[-2w_2' + \frac{d}{dM}\left(\frac{f}{t_s}\right) + \frac{f_x}{t_s}\{x_c' + w_2'(t_s - t_c) - w_2 t_c'\} \right] dM. \quad (39)$$

Here, $f(x,t)$ and $f_x(x,t)$ are evaluated at $x = x_c + (t_s - t_c)\, w_2$, $t = t_s$; M_i is the initial Mach number of the main shock wave. Equation (39) is solved by an iterative numerical technique; since for a fixed M the right-hand side of (39) increases monotonically with t_s and the left-hand side is a function of M alone, it is possible, for each increment in M, to solve for the corresponding t_s. Once t_s is determined, x_s is given by

$$x_s = x_c + w_2(t_s - t_c). \quad (40)$$

It should be noted that the secondary shock does not form at the initial point of the fluid flow field, $t = 0$, $x = 1$. By taking the limit $M \to M_i$ in (39), the time of shock formation t_i is obtained; its position is given by (40). If one neglects the expansion-region correction term f in (39), the initial time can be given explicitly as $t_i = (w_2 t_c' - x_c')/w_2'$; otherwise an iterative procedure must be used. For the problem given in the next section, the explosion of a sphere of gas at ambient temperature and a pressure ratio of 22, the point of shock formation was found to be $x_i = 1{\cdot}43$, $t_i = 1{\cdot}40$ when the correction term f was neglected; and $x_i = 1{\cdot}14$, $t_i = 0{\cdot}41$ for the full equations.

The technique described above, while useful to describe the initial motion of the secondary shock, is valid only when this shock is still weak. In the present situation, however, this shock strengthens as it is carried outward by the expanding gases and another method must be used to describe its subsequent motion. (The change-over point for a specific problem was arbitrarily chosen to be either when the shock begins to turn back toward the origin or when the shock strength becomes of magnitude unity, whichever comes first. Shock strength is defined in this case as $\bar{M} - 1$, where $\bar{M}$ is the secondary-shock Mach number.)

The method used in § 3, when altered to account for the non-uniform flow ahead of the secondary shock, is applicable for describing the later motion of this shock. In order to extend the formulation of § 3, we start with the characteristic differential relation corresponding to (20). Since the negative characteristics are the ones coming into the shock, we use

$$dp - \rho a\, du = -\frac{\rho a^2 u n}{u - a}\frac{dx}{x}. \quad (41)$$

The Hugoniot relations given in (21) are used with the variables a_0, p_0, ρ_0 replaced by a_3, p_3, ρ_3. Also, the velocity relation is altered to account for the fact that the shock is backward facing and the flow in region (3) is non-stationary, giving

$$u = u_3 - \frac{2a_3}{\gamma + 1}(\bar{M} - \bar{M}^{-1}),$$

where $\bar{M} = (u_3 - U_s)/a_3$, and U_s is the secondary shock velocity. Substituting these relations into (41), we obtain equations which hold along the secondary shock:

$$\left.\begin{aligned}\frac{dM}{dx} &= \left\{\frac{(\gamma+1)}{2}\frac{\bar{M}}{R}\left[\frac{u_{3t}/U_s + u_{3x}}{a_3}\right] - \left[\frac{\bar{M}^2-1}{R} + \frac{1}{\gamma-1}\right]\left[\frac{a_{3t}/U_s + a_{3x}}{a_3}\right]\right.\\ &\quad \left. + \left[\frac{2a_3(\bar{M}^2-1) - (\gamma+1)\bar{M}u_3}{(\gamma+1)\bar{M}u_3 - 2a_3(\bar{M}^2-1) - R}\right]\frac{n}{2x}\right\} \Big/ \left\{\frac{2\bar{M}}{2\gamma\bar{M}-\gamma+1} + \frac{\bar{M}^2+1}{\bar{M}R}\right\},\\ \frac{dt}{dx} &= U_s^{-1}, \quad U_s = u_3 - \bar{M}a_3;\\ \text{with}\qquad R &= \sqrt{\{(2\gamma\bar{M}^2-\gamma+1)(\bar{M}^2(\gamma-1)+2)\}}.\end{aligned}\right\} \tag{42}$$

The quantities u_3, a_3 and their derivatives are obtained from equations (11) and (16). Equations (42) are solved stepwise from a given point where x, t, and U_s or $\bar{M}$ are prescribed.

6. Application to a specific problem

We shall describe here the application of the present theory to a specific problem which has been solved theoretically by Brode (1957), and investigated experimentally by Boyer (1960). A sphere of radius 1, containing a gas at 22 atm. and at a temperature of 299° K, is assumed to be surrounded by air at 1 atm. and the same temperature. The compressed gas is also air, and the ratio of specific heats γ is assumed to equal 1·4 everywhere. The initial Mach number of the main shock, as determined by one-dimensional shock tube theory, is 1·846. With this, the constant on the right-hand side of the main shock equation (23) is 26·1. For this case the sound speed and particle velocity, made dimensionless with respect to the flow in region (0), are as follows at the initial point $x = 1, t = 0$:

$$\begin{aligned} u_0 &= 0, & a_0 &= 1,\\ u_1 &= 1{\cdot}087, & a_1 &= 1{\cdot}252,\\ u_2 &= 1{\cdot}087, & a_2 &= 0{\cdot}729,\\ u_3 &= \frac{5}{6}\left(1+\frac{x-1}{t}\right), & a_3 &= \frac{1}{6}\left(5-\frac{x-1}{t}\right),\\ u_4 &= 0, & a_4 &= 1.\end{aligned}$$

The leading characteristic of the expansion fan in region (3) is $(x-1)/t = -1$.

For region (3) we have, corresponding to equations (11), (16) and (17),

$$2{\cdot}5a_3 + 0{\cdot}5u_3 = 2{\cdot}5 - \frac{t}{28{\cdot}8}\left(5-\frac{x-1}{t}\right)\left(1+\frac{x-1}{t}\right)^2, \tag{6.11}$$

$$u_3 - a_3 = \frac{x-1}{t} - \frac{1}{72}\left[\left(5-\frac{x-1}{t}\right)\left(1+\frac{x-1}{t}\right)t\right]\left[\left(1+\frac{x-1}{t}\right) - \frac{12}{x-1}\left(1-\frac{\log x}{x-1}\right)\right], \tag{6.16}$$

$$x = 1 + Lt - \frac{1}{144}\left[\left(5-\frac{x-1}{t}\right)\left(1+\frac{x-1}{t}\right)t^2\right]\left[\left(1+\frac{x-1}{t}\right) - \frac{12}{x-1}\left(1-\frac{\log x}{x-1}\right)\right]. \tag{6.17}$$

The contact front is given by

$$\frac{dt_c}{dM} = \frac{4{\cdot}127\{x'_m + w'_1(t_c - t_m) - w_1 t'_m\}}{5a_3 + u_3 - w_1}, \tag{6.32}$$

$$x_c = x_m + w_1(M)(t_c - t_m). \tag{6.33}$$

Here $w_1(M)$ is the negative characteristic slope $u - a$ in region (1); it is evaluated, as a function of the main shock Mach number M, by using the Hugoniot relations in equation (21). In region (2) the negative characteristic slope w_2 is

$$w_2 = 0{\cdot}909 w_1 + 0{\cdot}182(2{\cdot}5a_3 + 0{\cdot}5u_3). \tag{6.29}$$

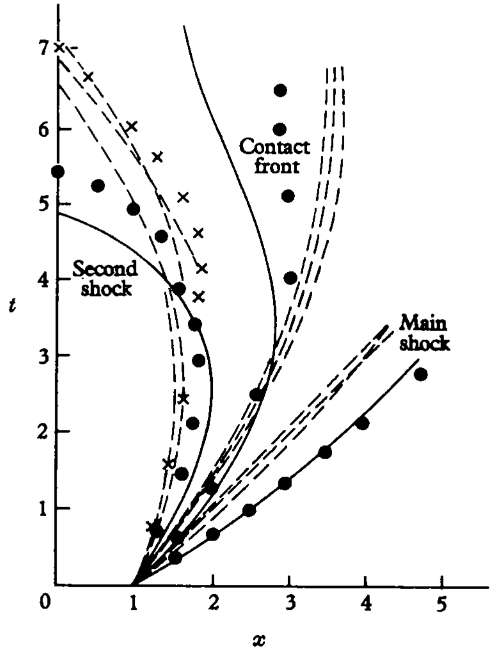

FIGURE 2. Experimental and theoretical spherical-blast results: ----, experiment (Boyer); ——, numerical integration (Brode); ●, present theory; ×, simplified second-shock approximation.

In figure 2 the present theory is compared with that of Boyer and Brode. The results of several experimental explosions carried out by Boyer are indicated by dashed lines. Brode's results, obtained by numerical integration of the Lagrangian equations of motion, are indicated by solid lines. The present theory is given by the curves composed of heavy dots. The time taken to compute the two shocks and the contact front on an IBM 704 was about 3 min.

Since Boyer's experimental work involved the bursting of a glass sphere, the differences between his result and Brode's may be due to the presence of glass particles in the flow field. As the present model, a hypothetical sudden expansion of a gas sphere, is the same as Brode's, it is expected that the results be comparable with his. The initial motion of the main shock and contact front as obtained by Brode compares quite favourably with that obtained with the present theory. It must be stated, however, that the present determination of the main shock is based entirely on Chisnell's work. Brode shows the secondary shock expanding slightly more, and imploding sooner, than both the present theory and Boyer's experiments. The differences in the initial motion could be due simply to the difficulty in interpreting data when the shock is weak. Brode, using the von Neumann–Richtmyer pseudo-viscosity technique, which spreads out the shock, might have difficulty in determining the true locations of a weak shock. Boyer's experimental technique, which used the schlieren shadowgraph to determine the secondary shock, would have difficulty in locating weak shocks which give correspondingly weak shadows. As to the determination of the time of implosion, Lagrangian mass co-ordinates, as used by Brode, may have lead to inaccuracies near the flow centre because the density there is quite low. Most of the mass is located near the contact front and main shock. However, since the present theory neglects certain entropy changes and in addition becomes less valid as the distance to the initial point ($x = 1$, $t = 0$) increases, it is rather difficult to determine precisely the causes of the small discrepancies in the different results. Over-all, however, the agreement is quite good.

A simplified determination of the secondary shock is indicated by the curve composed of $\times$'s in figure 2. For this, the one-dimensional solution was used to describe the flow in region (3). This is given by omitting the last term in equations (6.11), (6.16) and (6.17), and also setting f and f_x equal to zero in the secondary shock equation (39). This approximation for the secondary shock motion agrees remarkably well with Boyer's experimental work, and it may be useful for giving a qualitative description of the flow as it affords a marked simplification of some of the equations.

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